Differentialrechnung mit einer Variablen

Grenzwerte und Ableitungen

$$y' = f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon} = \frac{dy}{dx}$$

a) $y = C = const$, $y' = 0$

b) $y = ax + b$

$$y' = \lim_{\varepsilon \to 0} \frac{[a(x+\varepsilon)+b] - [ax+b]}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{a\varepsilon}{\varepsilon} = a$$

c) $y = x^2$

$$y' = \lim_{\varepsilon \to 0} \frac{(x+\varepsilon)^2 - x^2}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{x^2 + 2\varepsilon x + \varepsilon^2 - x^2}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\cancel{\varepsilon}(2x+\varepsilon)}{\cancel{\varepsilon}} = 2x$$

d) $y = \sqrt{x}$
$= x^{1/2}$

$$y' = \lim_{\varepsilon \to 0} \frac{\sqrt{x+\varepsilon} - \sqrt{x}}{\varepsilon} = \lim_{\varepsilon \to 0} \left(\frac{\sqrt{x+\varepsilon} - \sqrt{x}}{\varepsilon} \cdot \frac{\sqrt{x+\varepsilon} + \sqrt{x}}{\sqrt{x+\varepsilon} + \sqrt{x}} \right)$$

$$= \lim_{\varepsilon \to 0} \frac{\cancel{\varepsilon}\, 1}{\cancel{\varepsilon}\left(\sqrt{x+\varepsilon} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2} x^{-1/2}$$

e) Exponentialfunktion

$y = e^x$, $y' = e^x$

$y = e^{-x}$

$$y' = \lim_{\varepsilon \to 0} \frac{e^{-(x+\varepsilon)} - e^{-x}}{\varepsilon} = e^{-x} \lim_{\varepsilon \to 0} \frac{e^{-\varepsilon} - 1}{\varepsilon}$$

$$= e^{-x} \lim_{\varepsilon \to 0} \left(\frac{1}{e^{\varepsilon}} \cdot \frac{1 - e^{\varepsilon}}{\varepsilon} \right)$$

$$= e^{-x} \underbrace{\lim_{\varepsilon \to 0} \frac{1}{e^{\varepsilon}}}_{1} \cdot \underbrace{\lim_{\varepsilon \to 0} \frac{1 - e^{\varepsilon}}{\varepsilon}}$$

$$e \approx (1+\varepsilon)^{\frac{1}{\varepsilon}}$$

$$e^{\varepsilon} \approx 1 + \varepsilon$$

$$e^{\varepsilon} - 1 \approx \varepsilon$$

$$\frac{e^{\varepsilon} - 1}{\varepsilon} \approx 1 , \quad \lim_{\varepsilon \to 0} \frac{e^{\varepsilon} - 1}{\varepsilon} = 1$$

$$\lim_{\varepsilon \to 0} \frac{1 - e^{\varepsilon}}{\varepsilon} = -1$$

$$= -e^{-x}$$

$y = \sinh x$ $\quad y' = \cosh x$

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